



TITLE:

# Some remarks on the automorphic spectrum of the inner forms of $SL(N)$ (Automorphic Representations and Related Topics)

AUTHOR(S):

Li, Wen-Wei

---

CITATION:

Li, Wen-Wei. Some remarks on the automorphic spectrum of the inner forms of  $SL(N)$  (Automorphic Representations and Related Topics). 数理解析研究所講究録 2013, 1871: 65-74

ISSUE DATE:

2013-12

URL:

<http://hdl.handle.net/2433/195469>

RIGHT:

# Some remarks on the automorphic spectrum of the inner forms of $\mathrm{SL}(N)$

Wen-Wei Li

## Abstract

In this survey article, we start by reviewing Arthur's conjectures for the multiplicities of  $L^2$ -automorphic representations in the discrete spectrum. We also give a sketch of the main ideas thereof, as exemplified in Arthur's endoscopic classification for classical groups, and then discuss its relation with the Hiraga-Saito theory for the group  $\mathrm{SL}(N)$  and its inner forms. This is based on a talk given in the RIMS workshop "Automorphic Representations and Related Topics", Kyōto 2013.

## 1 Multiplicities in the discrete spectrum

Let  $F$  be a number field and  $\mathbb{A} := \mathbb{A}_F$  its ring of adèles. Fix an algebraic closure  $\bar{F}$  of  $F$ . We define  $\Gamma_F := \mathrm{Gal}(\bar{F}/F)$  and denote its Weil group by  $W_F$ . The Weil-Deligne group of  $F$  is denoted by  $W'_F$ .

For a connected reductive  $F$ -group  $G$ , one of the main concerns of the theory of  $L^2$ -automorphic forms is to study the right regular representation of  $G(\mathbb{A})$  on

$$L^2(G(F)\backslash G(\mathbb{A})^1) = L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1) \oplus (\text{continuous spectrum})$$

where  $G(\mathbb{A})^1$  is the kernel of the Harish-Chandra homomorphism  $H_G : G(\mathbb{A}) \rightarrow \mathfrak{a}_G$ .

It is known that the discrete part  $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1)$  decomposes into  $\bigoplus_{\pi} m(\pi)\pi$  with multiplicities  $m(\pi) < \infty$  for all  $\pi = \bigotimes'_v \pi_v$ . Our main goal is the study of  $m(\pi)$ . In this article, we adopt the usual convention that the archimedean components of  $\pi$  are viewed as Harish-Chandra modules. Assume hereafter:

- 1) the existence of the *automorphic Langlands group*  $L_F \rightarrow W_F$  (we shall write  $L'_F := W_F \times \mathrm{SU}(2)$ );

2)  $G$  is quasisplit.

The first assumption is of course too extravagant; we use it only to streamline the exposition. In particular, we can then talk about the A-parameters  $\psi : L'_F \rightarrow {}^L G := \hat{G} \rtimes W_F$ . The  $\hat{G}$ -conjugacy classes of A-parameters are expected to parametrize packets of automorphic representations of  $G(\mathbb{A})$ .

The internal structure of A-packets are expected to be controlled by the groups

$$\begin{aligned} S_\psi &:= \left\{ \hat{g} \in \hat{G} : \hat{g}\psi\hat{g}^{-1} = a \cdot \psi, \ a \in \ker^1(W_F, Z_{\hat{G}}) \right\}, \\ S_{\psi, \text{ad}} &:= S_\psi / Z_{\hat{G}}, \\ \mathcal{S}_\psi &:= \pi_0(S_{\psi, \text{ad}}, 1). \end{aligned}$$

The idea is that elements in  $\mathcal{S}_\psi$  gives rise to endoscopic data of  $G$  by which  $\psi$  factors through.

The group  $S_\psi \times L'_F$  acts on  $\hat{\mathfrak{g}} := \text{Lie}(\hat{G})$ , which gives a representation

$$\tau_\psi = \bigoplus_{\alpha} (\lambda_{\alpha} \boxtimes \mu_{\alpha} \boxtimes \nu_{\alpha}) \quad (\text{decomposition into irreducibles})$$

where the exterior tensor products are taken with respect to the product  $S_\psi \times L_F \times \text{SU}(2)$ . The relevance of these objects are explained as follows.

i) We define a sign character  $\varepsilon_\psi : \mathcal{S}_\psi \rightarrow \{\pm 1\}$  by setting

$$\varepsilon_\psi(x) := \prod_{\alpha} \det(\lambda_{\alpha}(s))$$

where  $s \in S_\psi$  projects to  $x \in \mathcal{S}_\psi$ , and the index  $\alpha$  ranges over those with  $\mu_{\alpha}$  symplectic and  $\varepsilon(\frac{1}{2}, \mu_{\alpha}) = 1$ .

ii) It is expected that to  $\psi$  is associated an A-packet  $\Pi_\psi$  of representations of  $G(\mathbb{A})$ , together with a map

$$\begin{aligned} \mathcal{S}_\psi \times \Pi_\psi &\rightarrow \mathbb{C}^\times, \\ (x, \pi) &\mapsto \langle x, \pi \rangle. \end{aligned}$$

iii) Set

$$m_\psi(\pi) := \frac{1}{|\mathcal{S}_\psi|} \sum_{x \in \mathcal{S}_\psi} \varepsilon_\psi(x) \langle x, \pi \rangle.$$

Now we can state Arthur's conjecture on the multiplicities [1].

*Conjecture 1.1.* For every admissible irreducible representation  $\pi$  of  $G(\mathbb{A})$ , we have

$$m(\pi) = \sum_{\psi} m_{\psi}(\pi)$$

where  $\psi$  ranges over the  $\hat{G}$ -conjugacy classes of A-parameters.

*Remark 1.2.* We note that in many cases (eg. the classical groups), this formula is expected to come from a decomposition into direct sums:

$$L_{\text{disc}}^2(G(F)\backslash G(A)^1) = \bigoplus_{\psi} L_{\psi}^2.$$

Consequently, every  $\pi$  in the discrete  $L^2$  spectrum should belong to at most one A-packet, say that corresponding to  $\psi$ , and we expect  $m(\pi) = m(\psi)$ .

## 2 Known cases

Arthur's conjectures are largely inspired by his study of the trace formula: see [3] for an excellent introduction. Here are a few known cases.

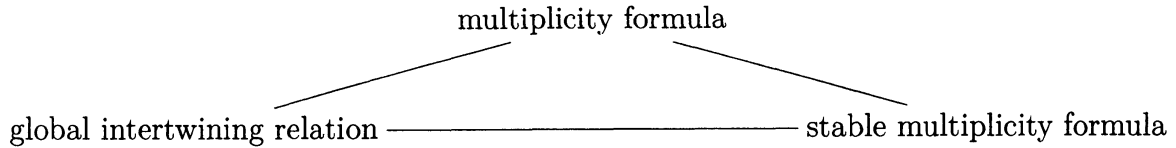
- A. For the quasisplit groups  $\text{SO}(2n+1)$ ,  $\text{Sp}(2n)$ , this is proved in [5], by using the selfdual irreducible cuspidal automorphic representations of  $\text{GL}(n)$  as a substitute for the A-parameters. In particular, there is no need to assume the existence of  $L_F$ . This is done by realizing these classical groups as elliptic endoscopic groups for the twisted space  $\widetilde{\text{GL}}(n)$ .
- B. For the quasisplit groups  $\text{SO}(2n)$ , a coarse version “up to outer automorphisms” is proved in [5], in which one can only identify the  $\text{O}(2n)$ -orbits of  $\psi$ .
- C. The case of  $U(3)$  is proved earlier by Rogawski [12].
- D. Arthur's machine is adopted to the quasisplit unitary groups  $U(n)$  by Chung Pang Mok [11]. There is no ambiguity of outer automorphisms.
- E. For the group  $\text{SL}(N)$ , Hiraga and Saito [8] have obtained the multiplicity formula for the generic spectrum by using the representations of  $\text{GL}(N)$  as substitutes of the A-parameters as before. They also obtained coarser results for the inner forms of  $\text{SL}(N)$ .

As regards the classical groups  $\text{SO}$ ,  $\text{Sp}$  and  $U$ , it would be interesting to consider the non-quasisplit cases as well, as alluded in [5, Chapter 9]. Some modifications of the definitions of  $S$ -groups are needed. The same remark certainly applies to  $\text{SL}(N)$  and its inner forms.

We will return to these issues later.

### 3 Arthur's approach

*Grosso modo*, Arthur's approach in [5, Chapter 4] can be summarized by the triad



in which any two terms imply the third one. The so-called stable multiplicity formula is a stable variant of our objective, the multiplicity formula. It pertains only to quasisplit groups. Note that in the Endoscopic Classification for classical  $G$ , these three properties are proved altogether in a long interlocking argument.

### 4 Stable multiplicity formula

Let  $S$  be a union of connected components of a reductive  $\mathbb{C}$ -group; these components generate a group  $\langle S \rangle$ , whose neutral component is denoted by  $S^\circ$ . Fix a maximal torus  $T$  in  $S^\circ$  and set

$$\begin{aligned}
 W^\circ &:= W(S^\circ, T), \\
 W &:= W(S, T) = N_S(T)/T.
 \end{aligned}$$

As usual, put  $\mathfrak{a}_T := \text{Hom}(X^*(T), \mathbb{R})$  and set

$$W_{\text{reg}} := \{w \in W : \det(w - 1)|_{\mathfrak{a}_T} \neq 0\}.$$

Fix a Borel subgroup of  $S^\circ$  containing  $T$ . For each  $w \in W$ , set

$$\varepsilon(w) := (-1)^{\#\{\alpha \in \Sigma(S^\circ, T) : \alpha > 0, w\alpha < 0\}}$$

where  $\Sigma(S^\circ, T)$  is the set of roots of  $(S^\circ, T)$ . We also write  $\varepsilon^G(w)$  to emphasize the ambient group  $G$ . The first goal is to “stabilize” the expression

$$i(S) := \frac{1}{|W^\circ|} \sum_{w \in W_{\text{reg}}} \varepsilon(w) |\det(w - 1)|^{-1}.$$

**Theorem 4.1.** *There exist unique constants  $\sigma(S_1)$  for each connected reductive  $\mathbb{C}$ -group  $S_1$ , such that*

- i)  $\sigma(S_1) = \sigma(S_1/Z_1)/|Z_1|$  for every central subgroup  $Z_1$ , this means in particular that  $\sigma(S_1) = 0$  if  $S_1$  is not semisimple;*

ii) for  $S$  as above, we have

$$i(S) = \sum_{\substack{s \in S/\text{conj} \\ \#Z(S_s^\circ) < \infty}} |\pi_0(S_s, 1)|^{-1} \sigma(S_s^\circ)$$

where  $S_s := Z_S(s)$ .

Assume hereafter in this section that  $G$  is quasisplit. By assuming the local Langlands correspondence and the endoscopic character relations, to each A-parameter  $\psi$  for  $G$  we may attach a stable distribution  $f \mapsto f(\psi)$  on  $G(\mathbb{A})$ . It satisfies  $f(\psi) = \prod_v f_v(\psi_v)$  if  $f = \prod_v f_v \in C_c^\infty(G(\mathbb{A})^1)$  and  $\psi_v$  is the local A-parameter deduced from  $\psi$ .

On the other hand, recall the stable trace formula for  $G$ , written as

$$I_{\text{disc}}^G(f) = \sum_{G'} \iota(G, G') S_{\text{disc}}^{G'}(f')$$

(cf. [2, §7]), where

- $I_{\text{disc}}^G$  is the discrete part of Arthur's *invariant trace formula* for  $G$ ;
- $G'$  ranges over the elliptic endoscopic data of  $G$ , identified somehow abusively with the associated endoscopic group;
- $\iota(G, G')$  are explicit positive constants;
- $f' \in C_c^\infty(G'(\mathbb{A}))$  is a Langlands-Shelstad transfer of  $f$ ;
- $S_{\text{disc}}^{G'}$  is the discrete part of the stabilized trace formula for  $G'$ , which is a stable distribution on  $G'(\mathbb{A})$ .

*Remark 4.2.* In Arthur's works, he has to introduce a parameter  $t > 0$  and consider the distributions  $I_{\text{disc}, t}^G$ , etc., to ensure absolute convergence. We deliberately omit this technical complication.

Now one can state the conjectural stable multiplicity formula.

*Conjecture 4.3.* For each  $f \in C_c^\infty(G(\mathbb{A})^1)$ , we have

$$S_{\text{disc}}^G(f) = \sum_{\psi} |\mathcal{S}_\psi|^{-1} \sigma((S_{\psi, \text{ad}})^\circ) \varepsilon^G(\psi) f(\psi).$$

As mentioned above, this is essentially proved for the groups  $\text{SO}$ ,  $\text{Sp}$  and  $U$ .

## 5 Intertwining relations

We consider only the local intertwining relation, as the global version [5, Corollary 4.2.1] is simply the product of its local avatars. Our main reference is [4].

Let  $F$  be a local field of characteristic zero,  $G$  be a connected reductive  $F$ -group and  $M$  a Levi subgroup of  $G$ . To each A-parameter  $\psi : W'_F \times \mathrm{SU}(2) \rightarrow {}^L G$ , we may define the groups  $S_\psi, \mathcal{S}_\psi$ . Moreover, if  $\psi$  factors through  ${}^L M \hookrightarrow {}^L G$ , say via  $\psi_M : W'_F \times \mathrm{SU}(2) \rightarrow {}^L M$ , then by assuming the local Langlands correspondence, we may form the A-packet  $\Pi_{\psi_M}$  of  $M$ .

Let  $\sigma \in \Pi_{\psi_M}$  and  $w \in N_G(M)(F)$  such that  $w\pi := \pi \circ \mathrm{Ad}(w^{-1})$  is isomorphic to  $\pi$ . Fix such an isomorphism  $\pi(w) : w\pi \xrightarrow{\sim} \pi$ . The variety  $Mw$  becomes a  $M$ -bitorsor under multiplication by  $M$ , as  $w$  normalizes  $M$ . That is,  $Mw$  is a twisted space in the sense of Labesse [10]. The assignment  $mwm' \mapsto \pi(m)\pi(w)\pi(m')$  gives rise to an irreducible representation of the twisted space  $Mw$  (see *loc. cit.*) Denote it by  $\pi_w$ .

Assume that  $\psi_M$  is invariant under the Weyl element associated to  $w$ . Then  $\psi_M$  can be plugged into the formalism of twisted endoscopy [9] for  $Mw$ . Define  $\Pi_{\psi_M}^w \subset \Pi_{\psi_M}$  to be the  $w$ -fixed elements in  $\Pi_{\psi_M}$ .

Let  $(M', s, \dots)$  be an elliptic endoscopic datum of the twisted space  $Mw$  by which  $\psi_M$  factors through via  $\psi' : W'_F \times \mathrm{SU}(2) \rightarrow {}^L M'$ . Consider a “lifting” of the elliptic endoscopic datum to  $G$ , upon replacing  $s$  by  $s' \in sZ_{Mw}^{\Gamma_F}/Z_{\hat{G}}^{\Gamma_F}$ :

$$\begin{array}{ccc} G' & \text{---} & (G, \text{inner twist by } \mathrm{Ad}(w)) \text{---} G \text{ untwisted} \\ \uparrow \text{Levi} & & \uparrow \text{Levi} \\ M' & \text{-----} & Mw \end{array}$$

where the dashed line means connection via elliptic endoscopic datum. We also assume that an L-embedding  ${}^L G' \hookrightarrow {}^L G$  is chosen.

*Conjecture 5.1.* Given a lifting as above, there exists a canonical map

$$\begin{array}{ccc} \Delta : & & \text{transfer factor for } (G', G) \\ \downarrow & & \\ \Delta_w : & & \text{twisted transfer factor for } (M', Mw), \end{array}$$

and there exist explicit constants  $c(\psi_{M,w})$  depending on the choice of an additive character  $\theta_F : F \rightarrow \mathbb{C}^\times$ , which should satisfy a global product formula, such that

$$f'(\psi') \rightarrow c(\psi_{M,w}) \sum_{\pi \in \Pi_{\psi_M}^w} \Delta_w(\psi'_w, \pi_w) \mathrm{tr}(R_P(\pi_w, \psi_M) I_P(\pi, f))$$

for all  $f \in C_c^\infty(G(F))$  where

- $\Delta_w(\psi'_w, \pi_w)$  is the spectral transfer factor corresponding to the geometric one  $\Delta_w$ ;
- $I_P(\pi)$  is the normalized parabolic induction with respect to a parabolic subgroup  $P = MU$ ;
- $R_P(\pi_w, \psi_M)$  is the normalized intertwining operator attached to  $\pi_w \in \Pi_{\psi_M}^w$  and  $\theta_F$ ;
- $f' \in C_c^\infty(G'(F))$  is a transfer of  $f$ .

Note that  $R_P(\pi_w, \psi_M)$  and  $\Delta_w(\psi'_w, \pi_w)$  depends on the choice of  $\pi(w) : w\pi \xrightarrow{\sim} \pi$ . But the ambiguities cancel with each other in the final expression. If  $G$  is quasisplit, we can normalize things by Whittaker models.

*Remark 5.2.* (a) For classical groups including the unitary groups, this conjecture can be simplified somehow and is proved in [5, 11]; note that the case of  $\mathrm{SO}(2n)$  is more delicate. (b) By taking  $M = G$  and  $w = 1$ , we revert to the endoscopic character formula for A-packets:

$$f'(\psi') = \sum_{\pi \in \Pi_\pi} \Delta(\psi', \pi) f(\pi)$$

where  $f(\pi) := \mathrm{tr} \pi(f)$ . (c) This local intertwining relation is used to construct general A-packets, as well as the relevant character identities, from the “elliptic” ones.

## 6 The work of Hiraga and Saito

The inner forms of  $\mathrm{SL}(N)$  serve as a reality check for Arthur’s conjectures. Let  $F$  be a local or global field of characteristic zero. Let  $D$  be a finite-dimensional central division algebra over  $F$ . Write

$$N = \dim_F D \cdot n$$

and consider

$$G^\sharp := \mathrm{SL}(n, D) \triangleleft \mathrm{GL}(n, D) =: G.$$

This construction yields all the inner forms of  $\mathrm{SL}(N, F) \triangleleft \mathrm{GL}(N, F)$ . A familiar technique for the study of representations of  $G^\sharp$  is to use the restriction from  $G$  to  $G^\sharp$ . The restriction ought be dual to the L-homomorphism  ${}^L G \twoheadrightarrow {}^L G^\sharp$  in view of the principle of functoriality. This is systematically done in [8], which we recall below.



When  $F$  is local, for every admissible irreducible representation  $\pi$  of  $G(F)$ , we define  $\Pi_\pi$  to be the set of irreducible constituents of  $\pi|_{G^\sharp(F)}$ . Note that  $\pi|_{G^\sharp(F)}$  is known to be semisimple of finite length. The finite sets  $\Pi_\pi$  are our candidates for the A-packets. For those  $\pi$  corresponding to a generic representation of  $\mathrm{GL}(N)$  via Jacquet-Langlands correspondence, Hiraga and Saito (a) related the internal structure of packets in terms of the  $\mathcal{S}$ -groups; (b) established the endoscopic character relations conjectured by Langlands.

When  $F$  is global, Hiraga and Saito studied the restriction of cusp forms. For cuspidal representations  $\pi = \bigotimes'_v \pi_v$  that are locally generic (up to Jacquet-Langlands correspondence), they derived a multiplicity formula à la Arthur, but with some undetermined constant in the non-quasisplit case. They made the assumption that  $G^\sharp$  is split at every archimedean place. Thanks to [6], this hypothesis is nowadays unnecessary.

One of the technical ingredients thereof is to reduce to the *automorphic induction* from  $\mathrm{GL}(N/d, E)$  to  $\mathrm{GL}(n, D)$ , where  $E/F$  is a cyclic extension of degree  $d$ . This reduction hinges on the seemingly folklore connection

$$\boxed{\text{Endoscopy of } G^\sharp} \longleftrightarrow \boxed{\text{Endoscopy of } G \text{ twisted by } \mathbf{a}, \text{ for various } \mathbf{a}}$$

where  $\mathbf{a}$  is an element in the continuous cohomology  $Z^1(W_F, Z_{\hat{G}})$  for  $F$  local (resp.  $\ker {}^1(W_F, Z_{\hat{G}})$  for  $F$  global). The latter box is exactly the case of automorphic induction for  $E/F$ , where  $E/F$  is the cyclic extension corresponding to  $\mathbf{a}$  by class field theory. The required endoscopic character identities then follow from those of automorphic induction by a “restriction” procedure for endoscopy.

It seems possible to verify Arthur’s conjectures using this formalism: one may try to formulate and verify

- the local intertwining relation for  $G^\sharp$  or its twisted variant for automorphic induction;
- the stable multiplicity formula for  $\mathrm{SL}(N)$ , which should be relatively easy.

The first obstacle is of course the extension of the local results in [8] to non-generic setting. The upshot is the character relation for automorphic induction of the *Speh representations*. Professor Hiraga has an unpublished proof for this using Zelevinsky involution (private communication). Granting this, it would be relatively easy to verify Arthur’s conjectures for  $G^\sharp = \mathrm{SL}(N)$  such as the stable multiplicity formula.

For the non-quasisplit case, it may help us to see the necessary modifications for Arthur’s conjectures in the non-quasisplit setting, such as the use

of modified  $S$ -groups, etc. For example, in the study of local intertwining relations, some phenomena unseen for classical groups might appear for the inner forms of  $SL(N)$ , cf. [7].

All these are obviously some immature thoughts. We hope to address the relevant issues in some future papers.

## References

- [1] J. Arthur. Unipotent automorphic representations: conjectures. *Astérisque*, (171-172):13–71, 1989. Orbites unipotentes et représentations, II.
- [2] J. Arthur. A stable trace formula. I. General expansions. *J. Inst. Math. Jussieu*, 1(2):175–277, 2002.
- [3] J. Arthur. An introduction to the trace formula. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 1–263. Amer. Math. Soc., Providence, RI, 2005.
- [4] J. Arthur. Induced representations, intertwining operators and transfer. In *Group representations, ergodic theory, and mathematical physics: a tribute to George W. Mackey*, volume 449 of *Contemp. Math.*, pages 51–67. Amer. Math. Soc., Providence, RI, 2008.
- [5] J. Arthur. The endoscopic classification of representations: Orthogonal and symplectic groups. preprint, 2012.
- [6] A. I. Badulescu and D. Renard. Unitary dual of  $GL(n)$  at Archimedean places and global Jacquet-Langlands correspondence. *Compos. Math.*, 146(5):1115–1164, 2010.
- [7] K. F. Chao and W.-W. Li. Dual R-groups of the inner forms of  $SL(N)$ . *ArXiv e-prints*, Nov. 2012. arXiv:1211.3039.
- [8] K. Hiraga and H. Saito. On  $L$ -packets for inner forms of  $SL_n$ . *Mem. Amer. Math. Soc.*, 215(1013):vi+97, 2012.
- [9] R. E. Kottwitz and D. Shelstad. Foundations of twisted endoscopy. *Astérisque*, (255):vi+190, 1999.
- [10] J.-P. Labesse. Stable twisted trace formula: elliptic terms. *J. Inst. Math. Jussieu*, 3(4):473–530, 2004.

- [11] C. P. Mok. Endoscopic classification of representations of quasi-split unitary groups. *ArXiv e-prints*, June 2012.
- [12] J. D. Rogawski. *Automorphic representations of unitary groups in three variables*, volume 123 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1990.

Wen-Wei Li

Institute of Mathematics,

Academy of Mathematics and Systems Science, Chinese Academy of Sciences,

55, Zhongguancun East Road,

100190 Beijing, China.

E-mail address: `wwli@math.ac.cn`